De Morgan's law and the Theory of Fields

Olivia Caramello and Peter Johnstone
DPMMS, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WB, U.K.

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Abstract

We show that the classifying topos for the theory of fields does not satisfy De Morgan's law, and we identify its largest dense De Morgan subtopos as the classifying topos for the theory of fields of nonzero characteristic which are algebraic over their prime fields.

Introduction

This note is a tailpiece to a recent paper [2] of the first author, in which necessary and sufficient conditions were given for the classifying topos of a geometric theory to satisfy De Morgan's law. A number of examples and counterexamples were given in that paper; one 'test case' which seemed worth considering was the (coherent) theory of fields, but it turned out that some additional ideas were needed to handle this case. Interestingly, the germ of these ideas was present in an old paper [3] of the second author — which happened to be published in the same volume as the first paper [4] in which the topos-theoretic ramifications of De Morgan's law were explored.

Another new result presented in [2] was the fact that, for every topos $\mathcal{E}$, there exists a largest dense subtopos of $\mathcal{E}$ satisfying De Morgan's law; we call this subtopos the *DeMorganization* of $\mathcal{E}$, by analogy with the Booleanization which is the largest (in fact only) dense Boolean subtopos. Explicit examples of DeMorganizations, for toposes which do not satisfy De Morgan's law, seem to be rather hard to find; but it turns out that the techniques of this paper give us such a description for the DeMorganization of the classifying topos for fields, and enable us to show that it classifies an easily described theory. These results are presented in section 2 of the paper; section 1 contains the proof that the classifying topos for fields does not itself satisfy De Morgan's law.

1 The Theory of Fields is not De Morgan

We recall that a commutative ring R is said to be (*von Neumann*) *regular* if, for every $x \in R$, there exists $y \in R$ satisfying $x^2y = x$ and $y^2x = y$. Note that this implies that R is nilpotent-free, since from $x^2 = 0$ we may deduce $x = x^2y = 0$. Also, the y whose existence is asserted is uniquely determined by x, since if y and z both satisfy the equations we have $x^2(y-z)^2 = (x-x)(y-z) = 0$, whence $x(y-z) = 0$, and therefore $y - z = (y^2 - z^2)x = x(y-z)(y+z) = 0$. Thus we may think of regular rings as commutative rings equipped with an additional unary operation $(-)^*$, satisfying $x^2x^* = x$ and $x(x^*)^2 = x^*$ for all x. (Note that it also follows from the uniqueness of x^* that we have $x^{**} = x$.)

Any field becomes a regular ring if we define $x^* = x^{-1}$ for all $x \neq 0$, and $0^* = 0$. Conversely, it is not hard to show that any prime ideal in a regular ring is maximal, and hence that any regular ring is a subdirect product of fields.

In what follows, we shall work with the category $\mathcal{C}$ of finitely-presented regular rings, considered as a full subcategory of the category **CRng** of commutative rings. (Note that any ring homomorphism between regular rings automatically commutes with the $(-)^*$ operation; similarly, if I is any (ordinary) ring ideal of a regular ring R, the quotient R/I is regular.) However, the reader should beware that finitely-presented regular rings are not in general finitely-presented as rings, because of the presence of the additional operation $(-)^*$.

We may define the notion of characteristic for regular rings, not as a single prime number but as a set of primes. For definiteness, let us write $\mathbb{P}$ for the set of (nonzero) prime numbers, and $\mathbb{P}^+$ for $\mathbb{P} \cup \{0\}$. Then we define

$$\text{Char } R = \{p \in \mathbb{P}^+ \mid \text{char } R/M = p \text{ for some maximal ideal } M \subseteq R\} \ .$$

Equivalently, $p \in \text{Char } R$ iff there exists a homomorphism from R to *some* field of characteristic p. (For nonzero p, we have the further equivalent condition that $p \in \text{Char } R$ iff $R/(p)$ is non-degenerate.) We note in passing that if there exists a homomorphism $h \colon R \to S$, then $\text{Char } S \subseteq \text{Char } R$; and we have equality here if h is injective, since if $k \colon R \to F$ is a homomorphism from R to a field, we can find a prime ideal of S disjoint from the set

$$\{h(x) \mid x \in R, k(x) \neq 0\} \ ,$$

and the quotient of S by this ideal must have the same characteristic as F.

Lemma 1.1 *If R is a finitely-presented regular ring, then* $\text{Char } R \subseteq \mathbb{P}^+$ *is either a finite set not containing* 0, *or a cofinite set containing* 0.

Proof First suppose there is some $n \in \mathbb{Z}^+$ such that $n.1 = 0$ in R. Then this equation holds in any field to which R can be mapped, so any such field has characteristic dividing n. Hence $\text{Char } R$ is contained in the (finite) set of prime divisors of n.

Otherwise, we have $n.1 \neq 0$ for all $n \in \mathbb{Z}^+$. Then the elements of this type form a multiplicatively closed set not containing 0, so we may find a prime (hence maximal) ideal M not meeting this set, and the quotient R/M is a field of characteristic 0. Thus $0 \in \text{Char } R$. Moreover, since R is finitely-generated as a regular ring, R/M must be finitely-generated as a field extension of $\mathbb{Q}$, so we can write it either as a pure transcendental extension $\mathbb{Q}(t_1, \ldots, t_d)$ or in the form $\mathbb{Q}(t_1, \ldots t_d, \alpha)$ where the t_i are independent transcendentals and α is algebraic over $\mathbb{Q}(t_1, \ldots, t_d)$. We shall deal with the second case; the first case is easier (as is the subcase $d = 0$ of the second).

We may suppose that the minimal polynomial for α over $\mathbb{Q}(t_1, \ldots, t_d)$ has the form

$$f_0(t_1, \ldots, t_d)\alpha^n + f_1(t_1, \ldots, t_d)\alpha^{n-1} + \cdots + f_{n-1}(t_1, \ldots, t_d)\alpha + f_n(t_1, \ldots, t_d) = 0$$

where the f_i are polynomials in the t_j with integer coefficients, and f_0 is not identically zero. At the cost, if necessary, of replacing α by an integer multiple of itself, we may further suppose that f_0 is primitive, i.e. that the highest common factor of its coefficients is 1. Now let H be the regular ring generated by $\{x_1, \ldots, x_d, y\}$ subject to the single equation

$$f_0(x_1, \ldots, x_d)y^n + f_1(x_1, \ldots, x_d)y^{n-1} + \cdots + f_n(x_1, \ldots, x_d) = 0 \ .$$

We claim that $\text{Char } H = \mathbb{P}^+$. It clearly contains 0, since the field R/M occurs as a quotient of H. And, for any prime p, we may choose values for $x_1, \ldots, x_d$ in some field extension of $\mathbb{Z}/(p)$ such that $f_0(x_1, \ldots, x_d) \neq 0$), and then choose a value for y in some algebraic extension of this field which satisfies the polynomial equation above; so we have a homomorphism from H to a field of characteristic p.

Now, for each finite subset F of $\mathbb{P}$ and each finite subset G of the set of all primitive polynomials in $\mathbb{Z}[x_1, \ldots, x_d]$, let $H_{F,G}$ be the quotient of H obtained by adding the relations $pp^* = 1$ for each

$p \in F$, and $g(x_1, \ldots, x_d)g(x_1, \ldots, x_d)^* = 1$ for each $g \in G$. By an easy extension of the argument above, $\text{Char } H_{F,G} = \mathbb{P}^+ \setminus F$ (note that forcing a primitive polynomial in the x_j to be invertible does not impose any restrictions on the characteristic). Moreover, it is clear that the $H_{F,G}$ form a directed diagram in $\mathcal{C}$, since we have a quotient map $H_{F,G} \to H_{F',G'}$ whenever $F \subseteq F'$ and $G \subseteq G'$, and that the colimit of this diagram in the category of all regular rings is isomorphic to R/M. But R, being finitely-presented, is finitely-presentable (in the categorical sense) as an object of the latter category; hence the quotient map $R \to R/M$ factors through $H_{F,G}$ for some pair (F,G). As we observed earlier, this forces $\text{Char } R \supseteq \text{Char } H_{F,G}$; so $\text{Char } R$ is cofinite. $\qquad\square$

Note in passing that no field of characteristic 0 can be finitely presented as a regular ring.

We now impose on $\mathcal{C}^{\text{op}}$ the Grothendieck topology J which makes $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J)$ into the classifying topos for the geometric theory of fields. As described in [5], D3.1.11(b), this is the smallest coverage for which the degenerate ring 0 is covered by the empty cosieve (we shall tend to think of the covers as cosieves in $\mathcal{C}$ rather than sieves in $\mathcal{C}^{\text{op}}$) and, for each object R and each $a \in R$, R is covered by the cosieve S_a^R generated by the two quotient maps $R \to R/(a)$ and $R \to R/(aa^* - 1)$. It is not hard to see that, since aa^* is idempotent, the induced map $R \to R/(a) \times R/(aa^* - 1)$ is an isomorphism, and hence that the coverage J is subcanonical; i.e. every representable functor $\mathcal{C}(S, -)$ is a sheaf for it. We shall also need:

Lemma 1.2 *For any J-covering cosieve S on an object R, we have $\text{Char } R = \bigcup\{\text{Char}(\text{cod } h) \mid h \in S\}$.*

Proof This is clearly satisfied by any of the generating cosieves S_a^R, since each homomorphism from R to a field factors through one of the two quotient maps generating it. We may now deduce the result by induction over the construction of J-covering cosieves; alternatively, we may observe that the cosieves satisfying the conclusion of the Lemma themselves form a Grothendieck topology on $\mathcal{C}^{\text{op}}$, which must therefore contain J. $\qquad\square$

Now, for any $A \subseteq \mathbb{P}^+$ and any $R \in \text{ob } \mathcal{C}$, let T_A^R denote the cosieve of all $h \colon R \to S$ for which $\text{Char } S \subseteq A$. By the previous lemma, T_A^R is a J-closed cosieve, so it defines a subobject of the representable functor $\mathcal{C}(R, -)$ in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J)$. Let us take R to be the initial regular ring I, and consider the cosieves T_A^I and T_B^I where A and B are complementary subsets of $\mathbb{P}$, both of them infinite.

Lemma 1.3 *Considered as subterminal objects in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J)$, the two cosieves just defined satisfy $\neg T_A = T_B$ and $\neg T_B = T_A$.*

Proof Clearly, $T_A \cap T_B$ contains only the morphism $I \to 0$, since 0 is the only regular ring whose set of characteristics is empty. But since 0 is covered by the empty cosieve, it represents the zero object of $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J)$. Hence $T_B \subseteq \neg T_A$. Moreover, a morphism $h \colon I \to R$ belongs to $\neg T_A$ iff it is stably disjoint from T_A; that is, if the only morphism $R \to S$ such that the composite $I \to R \to S$ belongs to T_A is the unique morphism $I \to 0$. But this implies that, for all $p \in A$, the quotient $R/(p)$ must be degenerate; so the nonzero members of $\text{Char } R$ must all lie in B. And since A is infinite, it follows from Lemma 1.1 that $0 \notin \text{Char } R$; so $\text{Char } R \subseteq B$ and $h \in T_B$. $\qquad\square$

Proposition 1.4 *The topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J)$ does not satisfy De Morgan's law.*

Proof With the notation of the previous lemma, it suffices to show that the join of T_A and T_B in the lattice of J-closed subterminal objects is not the top element $\mathcal{C}(I, -)$; equivalently, that the union $T_A \cup T_B$ is not J-covering. And this follows easily from Lemma 1.2, since we have $0 \in \text{Char } I$, but 0 does not occur in $\text{Char } R$ for any $I \to R$ in $T_A \cup T_B$. $\qquad\square$

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Recall that in [2], Theorem 3.1, a syntactic condition was given for the classifying topos of a geometric theory $\mathbb{T}$ to satisfy De Morgan's law. We may illustrate the failure of that condition explicitly in the present case: let ϕ be the formula (in the empty context) $\bigvee_{p \in A}(p.1 = 0)$. Then the condition would require the existence of geometric formulae ψ_1 and ψ_2 satisfying $(\top \vdash (\psi_1 \vee \psi_2))$, $((\psi_1 \wedge \phi) \vdash \bot)$, and such that $\chi \wedge \phi$ is consistent for every consistent χ satisfying $(\chi \vdash \psi_2)$. But the second of these conditions forces $(\psi_1 \vdash \bigvee_{p \in B}(p.1 = 0))$, since by Lemma 1.3 the disjunction on the right is the largest geometric formula inconsistent with ϕ, and hence the formula ψ_2 must be valid in any field of characteristic 0. It follows that there can be only finitely many primes p such that $(\psi_2 \wedge (p.1 = 0))$ is inconsistent; for if there were infinitely many such p, we could obtain a feld of characteristic 0 as an ultraproduct of fields of these characteristics — and although ψ_2 is not necessarily coherent, it can be written as a disjunction of coherent formulae ([5], D1.3.8), from which it follows that if $\neg\psi_2$ holds in each factor of an ultraproduct, it must also hold in the ultraproduct itself. Hence in particular, the formula $\chi = (\psi_2 \wedge (p.1 = 0))$ must be consistent for some $p \in B$; but then $(\chi \wedge \phi)$ is inconsistent.

Note that the crucial element in the above argument is the fact that the property of having characteristic zero is not definable by any geometric formula in the theory of fields. This was already observed in [3], where it was proved by a topological argument; it also follows from Lemma 1.1, since if the property were definable by a formula ϕ then the interpretation of ϕ in the generic field would be a J-closed cosieve on the initial regular ring I, consisting of morphisms $I \to R$ for which Char $R = \{0\}$.

2   The DeMorganization of the Theory of Fields

In this section, our aim is to identify the DeMorganization of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$, as defined in [2], 1.3. It is certainly of the form $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J')$ for some $J' \supseteq J$; moreover, we can identify at least some of the additional cosieves which J' must contain. In general, given any subobject $A' \rightarrowtail A$ in a topos $\mathcal{E}$, the inclusion $(\neg A' \vee \neg\neg A') \rightarrowtail A$ must be dense for the local operator correponding to the DeMorganization of $\mathcal{E}$, since it can be expressed as the pullback of $1 \amalg 1 \rightarrowtail \Omega_{\neg\neg}$ along the classifying map of $\neg A' \rightarrowtail A$. Hence, by the arguments presented in the previous section, the J-closed cosieve $T_{\mathbb{P}}$ must be J'-covering on the initial regular ring I, since it contains $T_A \cup T_B$. But the codomain of every morphism $h \colon I \to R$ in this cosieve has only a finite set of characteristics; and if Char $R = \{p_1, p_2, \dots, p_n\}$ then we may J-cover R by the cosieve generated by the quotient maps $R \to R/(p_i)$, $1 \leq i \leq n$. (For, by composing covers of the form $S_{p_i}^{R'}$, we may show that R is J-covered by the cosieve generated by these quotients together with $R \to R/(qq^* - 1)$, where q is the product of the p_i; but the latter quotient is degenerate because its characteristic set is empty.) Hence we see that the cosieve generated by the quotient maps $I \to I/(p)$, $p \in \mathbb{P}$, is J'-covering; equivalently, in the corresponding quotient theory of fields, the geometric sequent

$$(\top \vdash_{[]} \bigvee_{p \in \mathbb{P}}(p.1 = 0))$$

must be provable.

Now fix a (nonzero) characteristic p, and let $\mathbb{I}_p$ denote the set of all monic polynomials in $\mathbb{Z}[X]$ whose coefficients are all in the range $\{0, 1, \dots, p-1\}$ and which are irreducible as polynomials over $\mathbb{Z}/(p)$. (We include the linear polynomials $X + j$, $0 \leq j \leq p-1$.) We write $\mathbb{I}_p^+$ for $\mathbb{I}_p \cup \infty$; now if R is a regular ring with Char $R = \{p\}$ and $x \in R$, we define the *type* of x to be the subset of $\mathbb{I}_p^+$ defined by

- $f(X) \in \mathrm{Type}(x)$ iff there exists a homomorphism $h \colon R \to F$, F a field, such that $f(h(x)) = 0$; and

- $\infty \in \mathrm{Type}(x)$ iff there exists a homomorphism $h \colon R \to F$, F a field, such that $h(x)$ is transcendental over the prime field of F.

By arguments like those of Lemma 1.1, we may show that if R is finitely-presented as a regular ring, then the type of any $x \in R$ is either a finite set not containing ∞, or a cofinite set containing ∞. And

if we partition $\mathbb{I}_p$ into two disjoint infinite sets C and D, we may construct J-closed cosieves U_C and U_D on $I[x]/(p)$ (the quotient of the free regular ring on one generator x by the ideal of multiples of p), such that $h\colon I[x]/(p) \to R$ belongs to U_C iff $\mathrm{Type}(h(x)) \subseteq C$, and similarly for U_D. Just as before, we may show that each of U_C and U_D is the negation of the other as a subobject of $\mathcal{C}\,(I[x]/(p), -)$, and that their union is not J-covering; but it must be J'-covering. By composing with suitable J-covering cosieves as before, we deduce that $I[x]/(p)$ is J'-covered by the cosieve generated by all quotient maps $I[x]/(p) \to I[x]/(p, f(x))$, $f \in \mathbb{I}_p$.

In logical terms, this means that the corresponding theory must satisfy the sequent

$$((p.1 = 0) \vdash_x \bigvee_{f \in \mathbb{I}_p} (f(x) = 0)) \,,$$

and hence that it satisfies the sequent

$$((p.1 = 0) \vdash_x \bigvee_{f \in \mathbb{M}} (f(x) = 0))$$

where $\mathbb{M}$ is the set of all monic polynomials over $\mathbb{Z}$. Given the sequent established earlier, it follows that we have

$$(\top \vdash_x \bigvee_{f \in \mathbb{M}} (f(x) = 0)) \,,$$

or equivalently that $I[x]$ itself is covered by the cosieve generated by all quotient maps $I[x] \to I[x]/(f(x))$, $f \in \mathbb{M}$.

Thus we have shown that J' must contain the coverage (J'', say) which is generated by J together with the cosieves generated by $(I \to I/(p) \mid p \in \mathbb{P})$ and $(I[x] \to I[x]/(f(x)) \mid f \in \mathbb{M})$. We note that $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J'')$ is dense in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$, since these additional cosieves are stably nonempty. (However, J'' is not subcanonical: the induced map $I \to \prod_{p \in \mathbb{P}} I/(p)$ is not an isomorphism (since its domain is countable but its codomain is not), which says that $\mathcal{C}\,(I[x], -)$ does not satisfy the sheaf axiom for the first of the new covers.) To show that J' coincides with J'', it suffices to show that $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J'')$ satisfies De Morgan's law. First we need

Lemma 2.1 *The coverage J'' described above is* rigid *in the sense of* [5], *C2.2.18: that is, every object R has a smallest J''-covering cosieve, which is generated by the set of morphisms $R \to S$ for which S is* irreducible, *i.e. has no covering cosieves other than the maximal one.*

Proof We note first that every finite field occurs as an object of $\mathcal{C}$ (although, as we observed earlier, no field of characteristic zero does so), and they are irreducible for J'' since each J''-covering cosieve is generated by a family of quotient maps, and a field has no proper quotients. On the other hand, if R is any finitely-presented regular ring (with finite generating set $\{x_1, \ldots, x_n\}$, say), then by pushing out and composing copies of the two new covers we may show that the cosieve generated by all homomorphisms $h\colon R \to R/(p, f_1(x_1), \ldots, f_n(x_n))$, where p is a prime and each f_i is a monic polynomial, is J''-covering. The codomain (S, say) of such a morphism need not be a field; but it is a finite regular ring. So, if it is not already a field (or degenerate), it contains some nontrivial idempotent yy^* (where y is neither invertible nor zero), and the cosieve generated by the quotient maps $S \to S/(yy^*)$ and $S \to S/(yy^* - 1)$ is J-covering. Proceeding inductively, we deduce that S is J-covered by the cosieve generated by its quotient maps to fields, and hence R is J''-covered by the cosieve generated by its quotient maps to fields. $\square$

Corollary 2.2 $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J'')$ *satisfies De Morgan's law.*

Proof The previous lemma implies that it is equivalent to the functor category $[\mathcal{C}', \mathbf{Set}]$, where $\mathcal{C}'$ is the full subcategory of $\mathcal{C}$ whose objects are finite fields; and this category satisfies the Ore condition (cf. [5], D4.6.3(a)). $\square$

Thus we have proved

Proposition 2.3 *The DeMorganization of the classifying topos for the theory of fields is the classifying topos for the geometric theory of fields of finite characteristic, in which every element is algebraic over the prime field.*   □

To complete the story, we may also identify the Booleanization of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$ — equivalently, of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J'')$. Note first that since $\mathcal{C}'$ is the disjoint union of its subcategories corresponding to the different primes, $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J'') \simeq [\mathcal{C}', \mathbf{Set}]$ is a coproduct of open subtoposes corresponding to the primes, and its Booleanization is the coproduct of the Booleanizations of these subtoposes. So it suffices to fix a prime p, and consider the Booleanization of the topos $[\mathcal{C}'_p, \mathbf{Set}]$ where $\mathcal{C}'_p$ denotes the category of finite fields of characteristic p. Since this category satisfies the Ore condition, its double-negation coverage simply consists of all nonempty cosieves; that is, each finite field extension generates a covering cosieve, This means that the corresponding theory satisfies the sequents which say that every nonzero polynomial has a root; hence the Booleanization of $[\mathcal{C}'_p, \mathbf{Set}]$ classifies the theory of algebraically closed fields of characteristic p which are algebraic over $\mathbb{Z}/(p)$. By [5], C3.5.8, this topos is atomic (though not coherent; of course, coherence fails because of the infinite disjunction in the axiom saying that every element is algebraic). Hence, by arguments like those in [5], D3.4.10, it may be identified with the topos $\mathbf{Cont}(G_p)$ of continuous G_p-sets, where G_p is the group of automorphisms of the algebraic closure of $\mathbb{Z}/(p)$ (topologized, as usual, by saying that the pointwise stabilizers of finite subsets form a basis for the open subgroups). Thus we have

Proposition 2.4 *The Booleanization of the classifying topos for fields is the classifying topos for the theory of algebraically closed fields of finite characteristic, in which every element is algebraic over the prime field. Moreover, it is atomic over* $\mathbf{Set}$*; in fact it may be identified with the coproduct, over all primes p, of the toposes* $\mathbf{Cont}(G_p)$*.*   □

It is of interest to note that the (coherent) theory of algebraically closed fields of a given characteristic is complete but not $\aleph_0$-categorical (because a countable algebraically closed field may or may not contain transcendentals); hence, by the results of Blass and Ščedrov [1], the classifying topos of this theory is not Boolean. If we add the infinitary geometric axiom which says that there are no transcendentals, we get a theory which is '$\aleph_0$-categorical' (to the extent that this concept is meaningful for infinitary theories), and its classifying topos is not merely Boolean but atomic. It is also worth noting the following:

Corollary 2.5 *Let $\mathcal{E}$ be the classifying topos for the theory of algebraically closed fields of some fixed characteristic p. Then the DeMorganization of $\mathcal{E}$ coincides with its Booleanization.*

Proof First note that we may obtain a classifying topos for the theory of fields of characteristic p by cutting down the site $(\mathcal{C}^{\mathrm{op}}, J)$ to the full subcategory $\mathcal{C}_p$ of finitely-presented regular rings in which $p.1 = 0$ (and leaving the definition of the coverage unchanged). It is readily seen that the classifying topos for algebraically closed fields of characteristic p is a dense subtopos of this; hence, by [2], Proposition 1.5, its DeMorganization is simply its intersection with the DeMorganization of $\mathbf{Sh}(\mathcal{C}_p^{\mathrm{op}}, J)$. But the latter classifies the theory of fields of characteristic p in which every element is algebraic; so when we form the intersection we obtain the Booleanization of $\mathbf{Sh}(\mathcal{C}_p^{\mathrm{op}}, J)$.   □

The result of the Corollary remains true in characteristic 0, but the proof requires a little more work. We omit the details.

Finally, we make some remarks about the comparison between the (coherent) theory of fields and the theory considered in [3] under the name 'Diers fields'. The latter is the theory classified by the functor category $[\mathcal{D}, \mathbf{Set}]$ where $\mathcal{D}$ is the category of all fields which are finitely-generated over their

prime fields (the finitely-presentable objects in the category of fields). Since this category also satisfies the Ore condition, the classifying topos for Diers fields satisfies De Morgan's law. In [3] the second author gave a sketch of how to present the theory of Diers fields by adding appropriate new predicates and axioms to the coherent theory of fields, from which it follows that there is a canonical geometric morphism $u \colon [\mathcal{D}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$ corresponding to the forgetful functor from Diers fields to fields. This morphism is surjective (since every morphism $\mathbf{Set} \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$ factors through it); so it might be tempting to conjecture that it is the Gleason cover of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$ in the sense of [5], D4.6.8. However, this is not the case; since $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$ is a coherent topos and $[\mathcal{D}, \mathbf{Set}]$ is not compact ([3], 5.1), the geometric morphism between them cannot be proper ([5], C3.2.16(i)). Also, we have

Lemma 2.6 *The geometric morphism u defined above is not skeletal in the sense of* [5], D4.6.9.

Proof We recall that a morphism $f \colon \mathcal{F} \to \mathcal{E}$ is skeletal iff it maps $\mathbf{sh}_{\neg\neg}(\mathcal{F})$ into $\mathbf{sh}_{\neg\neg}(\mathcal{E})$. But we may identify the Booleanization of $[\mathcal{D}, \mathbf{Set}]$, as we did that of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$: since every field extension in $\mathcal{D}$ (including transcendental extensions) generates a covering sieve for the corresponding Grothendieck topology, it is easy to see that the points of $\mathbf{sh}_{\neg\neg}([\mathcal{D}, \mathbf{Set}])$ are the fields which are injective with respect to all morphisms of $\mathcal{D}$; hence they are exactly the algebraically closed fields (of any characteristic) which have infinite transcendence degree over their prime fields. (It is straightforward to use the additional predicates in the theory of Diers fields to express in geometric terms the statement that a field contains infinitely many independent transcendentals.) So u does not map the points of $\mathbf{sh}_{\neg\neg}([\mathcal{D}, \mathbf{Set}])$ into those of $\mathbf{sh}_{\neg\neg}(\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J))$. $\square$

We do not know whether u can be factored through the Gleason cover of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J)$. Since it is not skeletal, we cannot appeal to [5], D4.6.12 to construct such a factorization; and since $[\mathcal{D}, \mathbf{Set}]$ is not localic over $\mathbf{Set}$, we cannot appeal to the projectivity theorem ([5], D4.6.15) either.

References

[1] A.R. Blass and A. Ščedrov, Boolean classifying toposes, *J. Pure Appl. Alg.* 28 (1983), 111–140.

[2] O. Caramello, De Morgan classifying toposes (2008), arXiv:math.CT/0808.1519.

[3] P.T. Johnstone, A syntactic approach to Diers' localizable categories, in *Applications of Sheaves*, Lect. Notes Math. 753 (Springer–Verlag, 1979), 466–478.

[4] P.T. Johnstone, Conditions related to De Morgan's law, in *Applications of Sheaves*, Lect. Notes Math. 753 (Springer–Verlag, 1979), 479–491.

[5] P.T. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, vols. 1–2, Oxford Logic Guides 43–44 (Oxford University Press, 2002).